From planar graphs to embedded graphs - a new approach to Kauffman and Vogel's polynomial

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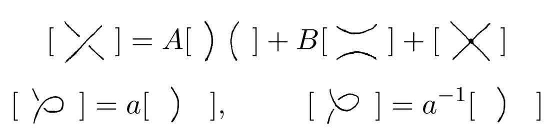
Abstract

In [4] Kauffman and Vogel constructed a rigid vertex regular isotopy invariant for unoriented four-valent graphs embedded in three dimensional space. It assigns to each embedded graph G a polynomial, denoted $[G]$, in three variables, A, B and a, satisfying the skein relations:

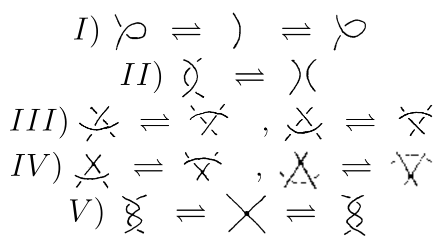


and is defined in terms of a state-sum and the Dubrovnik polynomial for links. Using the graphical calculus of [4] it is shown that the polynomial of a planar graph can be calculated recursively from that of planar graphs with less vertices, which also allows the polynomial of an embedded graph to be calculated without resorting to links. The same approach is used to give a direct proof of uniqueness of the (normalized) polynomial restricted to planar graphs. In the case $B = A^{-1}$ and $a = A$, it is proved that for a planar graph G we have $[G] = 2^{c-1}(-A - A^{-1})^v$, where c is the number of connected components of

G and v is the number of vertices of G. As a corollary, a necessary, but not sufficient, condition is obtained for an embedded graph to be ambient isotopic to a planar graph. In an appendix it is shown that, given a polynomial for planar graphs satisfying the graphical calculus, and imposing the first skein relation above, the polynomial extends to a rigid vertex regular isotopy invariant for embedded graphs, satisfying the remaining skein relations. Thus, when existence of the planar polynomial is guaranteed, this provides a direct way, not depending on results for the Dubrovnik polynomial, to show consistency of the polynomial for embedded graphs.

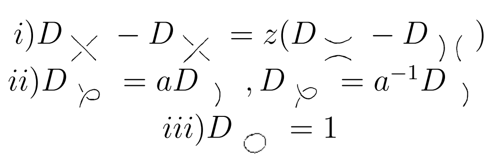
1 Introduction.

An *ambient isotopy* for 4-valent rigid vertex embedded graphs may be regarded [1] as a finite sequence of the following moves:



If the first move is not used in the sequence it is called a *regular isotopy.*

The *Dubrovnik polynomial* [2, 3] is the knot polynomial in 2-variables a, z invariant for regular isotopies that satisfies the axioms:



Kauffman and Vogel [4] construct a 3-variable polynomial for 4-valent rigid vertex embedded graphs which is invariant under regular isotopies, by using the skein relation [ ] = A[ ]+B[ ]+[ ] and the Dubrovnik polynomial with $z = A - B$.

It is constructed as follows. Let G be a 4-valent rigid vertex embedded graph diagram.

(1) Choose a marker or at each vertex of the graph.

(2) Let $\mathcal{L}$ be the set of links obtained by replacing:

$$\longmapsto \quad \text{or} \quad \text{or} \quad)\,($$
$$\longmapsto \quad \text{or} \quad)\,(\quad \text{or}$$

(3) Define $[G] = \sum_{L \in \mathcal{L}} (-A)^{i(L)}(-B)^{j(L)} D_L$ where $i(L)$ is the number of replacements of type $\longmapsto$, $j(L)$ is the number of replacements of type $\longmapsto)\,($ and D_L is the Dubrovnik polynomial of L with $z = A - B$.

Kauffman and Vogel [4] prove that this polynomial is well-defined and invariant under regular isotopy. It generalizes both the bracket polynomial (corresponding to the case $B = A^{-1}$ and $a = -A^3$) and the Yamada polynomial [5] (corresponding to the case $B = A^{-1}$ and $a = A^2$).

2 Graphical Calculus.

In [4] the following graphical calculus is proved.

Theorem 1 *For 4-valent graph diagrams, differing only in the marked local picture, we have the following identities:*

$$[\ )\bigcirc\ ] = \mu[\ )\ ]$$

$$[\ \ ] = \mathcal{O}[\ )\ ]$$

$$[\ \ ] = (1 - AB)[\ )(\ ] + \gamma[\ \ ] - (A + B)[\ \ ]$$

$$[\ \ ] - [\ \ ] = AB([\ \ ] - [\ \ ] + [\ \ ] - [\ \ ] + [\ \ ] - [\ \ ]) + \xi([\ \ ] - [\ \ ])$$

where

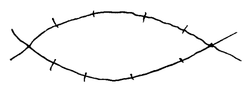
$$\mu = \frac{a - a^{-1}}{A - B} + 1$$
$$\mathcal{O} = \frac{Aa^{-1} - Ba}{A - B} - (A + B)$$
$$\gamma = \frac{B^2 a - A^2 a^{-1}}{A - B} + AB$$
$$\xi = \frac{B^3 a - A^3 a^{-1}}{A - B}$$

These identities allow us to calculate the polynomial of a planar 4-valent graph from the polynomials of other planar graphs with less vertices, if the graph contains one of the local configurations or or if it contains

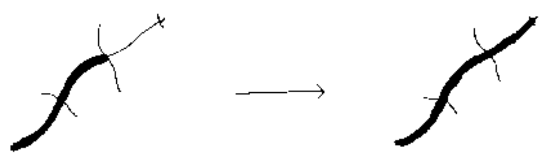
one of these configurations after a sequence of moves of the type  $\longrightarrow$  . This is in fact always the case, as the following result shows.

Lemma 2 *If a connected planar 4- valent graph has at least one vertex and does not contain either of the local configurations*  *or*  *then it is possible, by a sequence of moves of the type*  $\longrightarrow$  *, to turn it into a new planar 4-valent graph containing the local configuration*  *.*

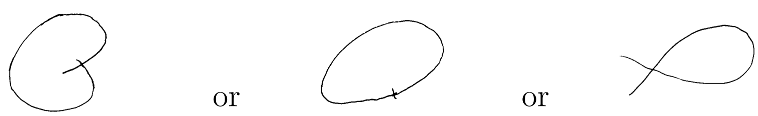
Proof. First we prove that the graph contains a global configuration of the type:



Let v be a vertex of the graph. We choose an edge that departs from v, and construct a walk which starts along this edge and carries on straight ahead at each vertex it meets. In other words, when the walk gets to a vertex it continues along the opposite edge, as opposed to turning to the left or right.

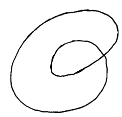


At a certain stage the walk returns to a vertex it has already visited producing one of the following three global configurations:

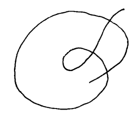


In the first case we prolong the walk so that we either obtain the third global configuration above, if the walk terminates or crosses itself before

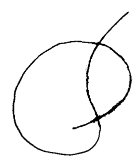
leaving the enclosed region:



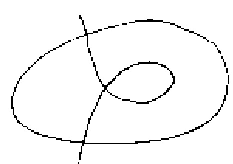
or



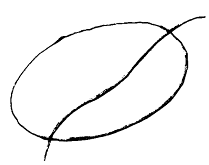
or the desired configuration, if the walk leaves the enclosed region without crossing itself before it does so:



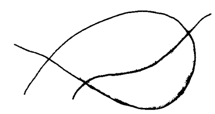
In the second case, starting at vertex v, we follow the transversal walk to the first walk into the enclosed region, and in this way we either obtain the third global configuration above:



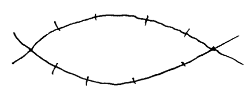
or the desired configuration:



In the third case, we take the set of all configurations of this type in the graph and choose one that is minimal (in the sense that it does not enclose another configuration of the same type). Since, by hypothesis, the graph does not contain loops, we can choose any walk that crosses this minimal configuration, and in this way get the desired configuration:



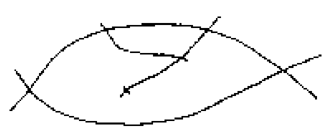
Now, since the graph contains configurations of the type



we can choose one that is minimal (in the sense that it does not contain another configuration of same type inside).

This configuration does not contain configurations of the second or third type inside it, since it is minimal. Thus there are no walks starting inside the configuration which do not eventually cross to the outside, and any walk that goes in through one of the sides of the configuration goes out through the other side without intersecting itself.

Suppose, first, that the configuration has vertices inside it. Start a walk at one of these vertices and extend it until it crosses to the outside. At the last vertex before this walk crosses to the outside, take the transversal walk that leaves the configuration through the same side.

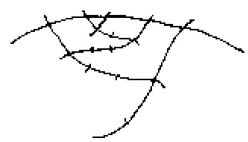


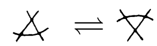
Thus, we get a global triangle adjacent to one side of the configuration.

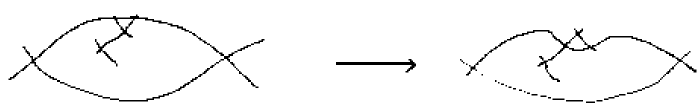
At the last vertex of the second walk before it crosses to the outside, take the transversal walk that goes into the triangle. This walk must leave the triangle through the side of the configuration, because if the walk went out through the same side of the triangle as it went in, the configuration would

not be minimal, and the third side has no vertices available for it to cross.

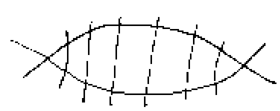
We iterate this last step so as to obtain a finite chain of triangles $T_1 \supset T_2 \supset ... \supset T_n$, all adjacent to one side of the configuration.

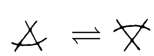
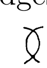


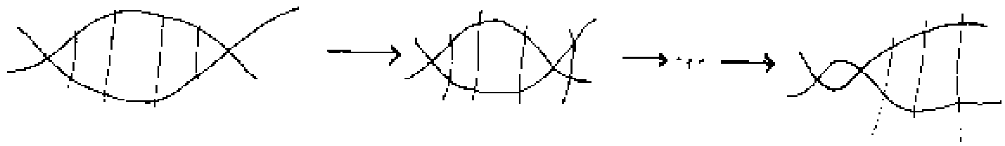
The last of these is a simple triangle.Thus, we can remove one vertex from inside the configuration by a move of the type  :



Applying this method recursively we can remove all vertices from inside the configuration, which takes on the following form:



Then we can apply the move  successively removing all edges inside the configuration. At the end, we obtain the desired configuration  (with no edges or vertices inside).



■

With this graphical calculus we can calculate the polynomial of planar graphs without resorting to links. We note that, since the polynomial is invariant under regular isotopies, the position of the various connected components of the graph does not matter: e.g. [ ] = [ ] . In fact, we have the following result:

Theorem 3 *There is a unique polynomial for planar 4-valent graphs that satisfies the graphical calculus and takes value 1 for the unknot.*

Proof. Let [.] and $[.]^*$ be two polynomials satisfying the conditions of the theorem. We will prove, by induction on the number of vertices, that $[G] = [G]^*$ for any planar 4-valent graph G.

If G has no vertices then G is an disjoint union of unknots, and hence, by the first identity of graphical calculus, $[G] = [G]^* = \mu^{c-1}$ where c is the number of unknots in G.

By the induction hypothesis, we suppose that [.] and $[.]^*$ are equal for graphs with up to n vertices.

Let G be a graph with $n+1$ vertices.

If G contains a local configuration of the type (or) then, by the second (or third) identity of the graphical calculus and the induction hypothesis, we get $[G] = [G]^*$.

Otherwise, by lemma 2 it is possible to produce a sequence of graphs $G \longrightarrow G_1 \longrightarrow .... \longrightarrow G_m$ using moves of the type $\rightleftharpoons$ where G_m contains a configuration of the type . Then, using the last identity of the graphical calculus and the induction hypothesis, we conclude that $[G] - [G_1] = [G]^* - [G_1]^*$, $[G_1] - [G_2] = [G_1]^* - [G_2]^*$, ...,$[G_{m-1}] - [G_m] = [G_{m-1}]^* - [G_m]^*$, and hence $[G] - [G_m] = [G]^* - [G_m]^*$. Now $[G_m] = [G_m]^*$ because G_m contains a configuration of the type , and thus we conclude that $[G] = [G]^*$. ■

Moreover, we can calculate the polynomial of a 4-valent rigid vertex embedded graph using this calculus in the following way:

Let G be a 4-valent rigid vertex embedded graph. Let $\mathcal{P}$ be the set of planar graphs obtained by replacing

$$\times \longmapsto)\,(\;,\; \asymp \; \textit{or} \; \times$$

Then $[G] = \sum_{P \in \mathcal{P}} A^{i(P)} B^{j(P)} [P]$ where $i(P)$ is the number of replacements of type $\times \longmapsto)\,($ and $j(P)$ is the number of replacements of type $\times \longmapsto \asymp$.

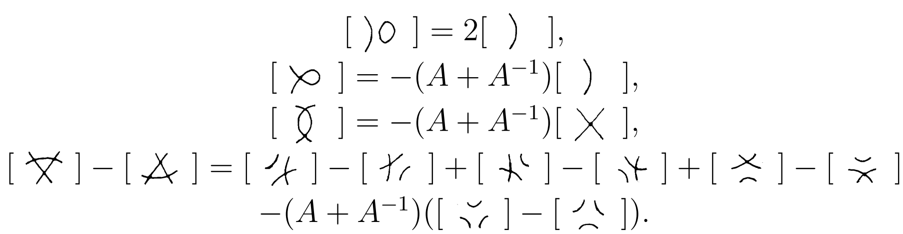
It is also possible to show that the invariance of the polynomial under regular isotopies is a consequence of the identities of theorem 1 (see appendix).

3 The case when $B = A^{-1}$ and $a = A$.

If we make the choice $a = A$ and $B = A^{-1}$ then:

$$\begin{gathered} \mu = 2, \\ \mathcal{O} = -A - A^{-1}, \\ \gamma = 0, \\ \xi = -A - A^{-1}. \end{gathered}$$

In this case the graphical calculus takes the form:



Theorem 4 *In the case $a = A$ and $B = A^{-1}$ for any 4-valent planar graph G we have $[G] = 2^{c-1}(-A - A^{-1})^v$, where c is the number of connected components of G and v is the number of vertices of G.*

Proof. By Theorem 3 we only have to prove that $[\bigcirc] = 1$ (that is obvious) and that the polynomial $2^{c-1}(-A-A^{-1})^v$ satisfies the graphical calculus with $B = A^{-1}$ and $a = A$.

The three first identities are evidently satisfied.

Now, let us look at the last identity. This identity is invariant under the 6-dihedral group D_6. If we remove from the graphs in the equation the configurations in which they differ and label the free ends 1 to 6 we have, up to the symmetry of the equation, the following distinct cases:

1. $1, 2, 3, 4, 5, 6$ are in the same connected component  ;

2. $1, 2$ are in one connected component and $3, 4, 5, 6$ in another  ;

3. The free ends group into connected components as: $\{1, 2\}, \{3, 4\}, \{5, 6\}$  ;

4. The free ends group into connected components as: $\{1, 2\}, \{3, 6\}, \{4, 5\}$  .

In case 1 we have [ ] = [ ], [ ] = [ ], [ ] = [ ], [ ] = [ ] and [ ] = [ ], and thus the identity is satisfied.

In case 2 we have [ ] = [ ], [ ] = [ ], [ ] = [ ], 2[ ] = [ ] and [ ] = 2[ ], and thus the identity is $0 =$ [ ] $-$ $(-A - A^{-1})$[ ] $= (-A - A^{-1})$[ ] $- (-A - A^{-1})$[ ] $= 0$, i.e. the identity is satisfied.

In case 3 we have [ ] = [ ], 2[ ] = [ ], 2[ ] = [ ], 2[ ] = [ ] and [ ] = 4[ ], and thus the identity is $0 =$ [ ] $+$ [ ] $+$ [ ] $-3(-A-A^{-1})$[ ] $= 3(-A-A^{-1})$[ ] $-3(-A-A^{-1})$[ ] $= 0$, i.e. the identity is satisfied.

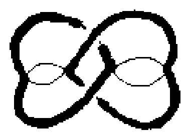
In case 4 we have [ ] = [ ], [ ] = [ ], [ ] = [ ], [ ] = [ ] and [ ] = [ ], and thus the identity is satisfied. ■

If we replace the vertices of a graph G by crossings we obtain a link L . A *knot-theoretic circuit* of G is a closed walk on G which corresponds to a link component of L. We define the *writhe* of a knot-theoretic circuit D as $w(D) = \sum\limits_{c \in C(D)} \epsilon(c)$ where $C(D)$ is the set of self-crossings of D and $\epsilon(c)$ is the sign (± 1) of the crossing c for a chosen orientation (and independent of this choice). The *twisting number* $t(G)$ of G is the sum of the writhes over all knot-theoretic circuits of G.

Corollary 5 *A necessary condition for a 4-valent rigid vertex embedded graph G be isotopic to a planar graph is that the polynomial of G with $B = A^{-1}$ and $a = A$ is $[G] = 2^{c-1}(-A - A^{-1})^v A^{t(G)}$ where c is the number of connected components, v is the number of vertices and $t(G)$ is the twisting number of G.*

This corollary is a consequence of the fact that the polynomial $a^{-t(G)}[G]$ is an isotopy invariant.

Unfortunately, this condition is not sufficient as we see in the following example.

G :  $c = 1, t(G) = 2, v = 4.$

Example 1 *The graph in the picture is not planar, since it contains two disjoint linked cycles (shown bold in the picture), but its polynomial is $A^2(-A - A^{-1})^4 = 2^{c-1}A^{t(G)}(-A - A^{-1})^v$.*

Corollary 6 *If a graph diagram G has only one crossing, and the removal of this crossing does not change the number of connected components, then the polynomial of G vanishes in the case $B = A^{-1}$ and $a = A$, and thus G is not isotopic to a planar graph.*

Proof. $[G] = A[G_1] + A^{-1}[G_2] + [G_3]$ where G_1 is the graph obtained by the change $\longrightarrow$)(, G_2 is the graph obtained by the change $\longrightarrow$ and G_3 is the graph obtained by the change $\longrightarrow$. Since the number of connected components of G does not change when the crossing

is removed, G_1,G_2 and G_3 have the same number of connected components, and therefore $[G_1] = [G_2]$ and $[G_3] = (-A - A^{-1})[G_1]$. Thus $[G] = (A + A^{-1})[G_1] + (-A - A^{-1})[G_1] = 0$. ■

4 Comments.

We can prove that the graphical calculus implies the invariance of the polynomial for regular isotopies (see the appendix). Thus, if we can prove the consistence of the graphical calculus without using the invariance of the polynomial, then we don't need to use the Dubrovnik polynomial to prove the invariance of the polynomial. This is achieved in the case $B = A^{-1}$ and $a = A$ because Theorem 4 gives a polynomial that is consistent with the graphical calculus, and which is unique by Theorem 3.

We can also see that this choice of variables ($B = A^{-1}$ and $a = A$) is the most interesting case, assuming that for planar graphs the polynomial is of the form $[G] = p^{c-1}q^v$, where p and q are two polynomials, c is the number of connected components and v is the number of vertices of G. In fact, by the two first identities of the graphical calculus we get $p = \mu$ and $q = \mathcal{O}$. By a reasoning analogous to the proof of Theorem 4, the third identity gives us the following equations:

$$\mathcal{O}^2 = 1 - AB + \gamma - (A+B)\mathcal{O}$$
$$\mathcal{O}^2 = (1 - AB)\mu + \gamma - (A+B)\mathcal{O}$$
$$\mathcal{O}^2 = 1 - AB + \gamma\mu - (A+B)\mathcal{O}$$

If $\mu \neq 1$ then we have $AB = 1$ and $\gamma = 0$, thus $a = A$ or $a = -A^3$. The second case implies $\mathcal{O} = 0$, and thus we get a generalization of the bracket polynomial that vanishes for graphs with at least one vertex. If $\mu = 1$ it is easy to prove, by induction on the number of crossings, that for any 4-valent rigid vertex embedded graph G we have $[G] = (A+B+\mathcal{O})^{cr}\mathcal{O}^v$, where cr is the number of the crossings of the embedding and v is the number of vertices of G, and we also have $A + B + \mathcal{O} = a$ and $a = a^{-1}$. Thus, in this case, the invariant $a^{-t(G)}[G]$ is almost trivial: $a^{-t(G)}[G] = \mathcal{O}^v$ if $t(G)$ and cr have the same parity (both even or both odd) and $a^{-t(G)}[G] = a\mathcal{O}^v$ otherwise.

Acknowledgment - I wish to thank Prof. Roger Picken who encouraged me to write this paper and helped me to improve it.

A Appendix.

We show that the graphical calculus implies the regular isotopy invariance of the polynomial.

Proposition 7 *If $[G]$ is a polynomial for 4-valent graph diagrams that satisfies the graphical calculus, and the skein relation* $[\ ] = A[\ ] + B[\ ] + [\ ]$ *holds, then it is invariant under regular isotopies for 4-valent rigid vertex embedded graphs and satisfies the identities* $[\ ] = a[\ ]$ *and* $[\ ] = a^{-1}[\ ]$.

Proof. 1. $[\ ] = A[\ ] + B[\ ] + [\ ]$
$= A\mu[\ ] + B[\ ] + \mathcal{O}[\ ]$
$= a[\ ]$
and $[\ ] = A[\ ] + B[\ ] + [\ ]$
$= A[\ ] + B\mu[\ ] + \mathcal{O}[\ ]$
$= a^{-1}[\ ]$
, since $\mathcal{O} + A\mu + B = a$ and $\mathcal{O} + A + B\mu = a^{-1}$.

2. To prove invariance under regular isotopy, it is enough to show the invariance of the polynomial under the generalized Reidemeister moves $II) - V)$.

II) $[\ ] = A[\ ] + B[\ ] + [\ ]$
$= Aa^{-1}[\ ] + B[\ ] + [\ ]$
$= Aa^{-1}[\ ] + BA[\ ] + B^2[\ ] + B[\ ] + A[\ ] + B[\ ] + [\ ]$
$= AB[\ ] + (Aa^{-1} + B^2 + B\mathcal{O})[\ ] + (A + B)[\ ] + [\ ]$
$= AB[\ ] - \gamma[\ ] + (A+B)[\ ] + [\ ] = [\ ]$, by the third identity of the graphical calculus.

IV) $[\ ] = A[\ ] + B[\ ] + [\ ] = A^2[\ ] + AB[\ ] + A[\ ] + BA[\ ] + B^2[\ ] + B[\ ] + A[\ ] + B[\ ] + [\ ]$
$= A^2[\ ] + AB[\ ] + A[\ ] + BA\mathcal{O}[\ ] + B^2[\ ] + B(1-AB)[\ ] + B\gamma[\ ] - B(A+B)[\ ] + A(1-AB)[\ ] + A\gamma[\ ] - A(A+B)[\ ] + B[\ ] + [\ ]$
$= -AB([\ ] - [\ ] + [\ ]) + (BA\mathcal{O} + A\gamma + B\gamma)[\ ] + B(1-AB)[\ ] + A(1-AB)[\ ] + A[\ ] + B[\ ] + [\ ]$
$= -AB([\ ] - [\ ] + [\ ]) + \xi[\ ] + B(1-AB)[\ ] + A(1-AB)[\ ] + A[\ ] + B[\ ] + [\ ]$.

Rotating the equation by 180^o we obtain:

$[\ ] = -AB([\ ] - [\ ] + [\ ]) + \xi[\ ] + B(1-AB)[\ ] + A(1-AB)[\ ] + A[\ ] + B[\ ] + [\ ]$. Note that the skein relation is invariant under rotations.

Thus $[\ ] - [\ ] = -AB([\ ] - [\ ] + [\ ] - [\ ] + [\ ] - [\ ]) - \xi([\ ] - [\ ]) + B(1-AB)([\ ] - [\ ]) + A(1-AB)([\ ] - [\ ]) + A([\ ] - [\ ]) + B([\ ] - [\ ]) + [\ ] - [\ ] = 0$.

III) $[\ ] = A[\ ] + B[\ ] + [\ ]$
$= A[\ ] + B[\ ] + [\ ] = [\ ]$.

V) $[\ ] = -A[\ ] - B[\ ] + [\ ]$
$= -A[\ )(\ ] - B[\ ] + [\ ] = [\ ]$. ■

Since the polynomial of an embedded graph can be given as a weighted sum of polynomials of planar graphs, we only need to assume that the graphical calculus holds for planar graphs.

References

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